Buckling Analysis of Orthogonally Stiffened Waffle Cylinders

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The energy expressions are formulated for the orthogonally stiffened waffle cylinder. The principle of minimum potential energy is employed to formulate the buckling equations for a simply supported waffle cylinder under both uniform and bending stresses. A variety of application examples are demonstrated. For the case of uniform compression, an optimum analysis of the buckling strength of waffle cylinder with constant weight is performed. The effect of the parameters such as the thickness of the shell, the ratio between the stiffener depth to the stiffener width, and the width-ratio between the stringers to the rings are studied. Suggestions for choosing such parameters for design are given. For the case of bending load, it is found that considerable buckling strength can be gained with wall stiffening without increasing the total weight. It is also found that for the present example the maximum buckling bending stress is 1.075 times of the uniform buckling stress.

Nomenclature

a	= spacing between the stiffeners
d	= depth of the stiffeners
E	= modulus of elasticity
h	= thickness of the shell skin
L	= length of the cylindrical shell
m, n	= longitudinal half-wave and circumferential full-wave numbers, respectively
N_c	= uniform axial stress
N_{co}	= uniform critical axial stress
N_b	= maximum bending stress
N_{bo}	= maximum critical bending stress in the absence of uniform axial stress
r	= mean radius of the skin of cylindrical shell
t_1, t_2	= widths of the longitudinal and circumferential stiffeners, respectively
u, v, w	= displacements in the x, θ, and z directions, respectively
U_{mn}, V_{mn}, W_{mn}	= displacement amplitudes
x, θ, z	= cylindrical coordinates
$\varepsilon_1, \varepsilon_2, \varepsilon_{12}$	= direct and shearing strain components
$\sigma_1, \sigma_2, \sigma_{12}$	= direct and shearing stress components
ν	= Poisson's ratio

Introduction

ONE of the most common structural elements in the modern airplane, missile, booster, and other aerospace vehicles is the thin circular cylindrical shell. The thin shell is weak in resisting the oftenly encountered high compressive stresses. The design of thin shells with high buckling strength-to-weight ratio is thus of major concern to the structural engineers.

One of the common means to increase the buckling strength of the shell is to stiffen the shell by the use of longitudinal stringers and circumferential rings. In this paper, an integrally stiffened waffle shell with stringers and rings, as shown in Fig. 1, is studied. The effects of the stringers and rings on the buckling strength are analyzed by the use of the principle of minimum potential energy.

In the method of minimization of potential energy, the three buckling displacements generally are represented by the double summations of a sufficiently large number of the products of double trigonometric functions and the corresponding amplitudes which satisfy the boundary conditions. Such displacement functions are then substituted into the potential energy expressions. Upon the execution of the first variation on the total potential energy, a set of stiffness matrix equilibrium equations may be obtained. Upon the execution of the second variation on the total potential energy or vanishing the stiffness matrix, the state of neutral stability is obtained which yields as many eigenvalues as the number of the amplitude terms. The lowest eigenvalue yields the critical buckling load and the corresponding eigenvectors describe the buckling mode shapes. This method requires the solution of eigenvalues of a large matrix of order $3kl$ with k and l being the numbers of terms retained in each of the series displacement functions. This method has been used previously by Meyer[1] for solving buckling loads of 45° eccentrically stiffened waffle shells (using one term in the displacement functions, i.e., $k = 1$ and $l = 1$) and then by Hofmeister and Felton[2,3] in the solution of buckling loads and optimum design of waffle plate with multiple rib sizes.

By using the method of energy minimization, the buckling problem of a waffle cylinder with orthogonally oriented stiffeners is formulated. The formulation is then used to perform a study of the buckling strength of such stiffened shell systems.

The first case is the study of a simply supported waffle cylinder subjected to uniform axial compression. Because of the effect of the smoothing of the stiffeners over the shell surface, the buckling mode of the cylinder in such case becomes the same as those represented by a single double trigonometric function, as in the classical case of the buckling of the monocoque cylinder.[4] The stiffness matrix reduces to the order of 3 by 3. By varying the longitudinal and circumferential wave numbers, the eigenvalues can be found and plotted as festoon curves. The lowest value yields the buckling load and the corresponding wave numbers give the mode shape. Meyer[1] has used this

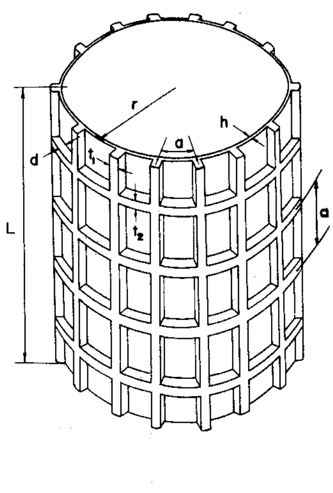

Fig. 1 An orthogonally stiffened waffle cylindrical shell.

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method to investigate the buckling loads of simply supported waffle cylinders with stiffeners oriented at $\pm 45°$ directions with the generator. Experimental verification was also carried out for every buckling load calculated.

Since the stiffness matrix is only of order 3×3 which is relatively easy for computation, it is used to analyze the optimum buckling loads of the simply supported and orthogonally-stiffened waffle cylinder with a constant weight. The parameters such as the thickness of the shell, the ratio between the stiffener depth to the stiffener width, the ratio between the width of the stringer (longitudinal stiffener) to the width of the ring (circumferential stiffener) are varied in the study. The general trends and the effects of such parameters on the buckling strength are discussed.

The second case is the study of a simply supported orthogonally stiffened waffle cylinder under combined bending and axial loads. Since the axial stress varies in the circumferential direction, the circumferential mode shape must be represented by the summation of a series of geometric function instead of a single function. This case is, however, another special case of the general formulation that the mode shape in both the longitudinal and the circumferential directions are represented by summations of series functions.

To insure the validity of the present formulation, a special case of a monocoque cylinder under pure bending is checked for which the buckling load can also be obtained by the use of an alternative Galerkin's method combined with Batdorf's modified version of Donnell's equation.[5] The buckling load of the waffle cylinder due to bending is then analyzed, and the mode shapes are presented. The advantage of the waffle cylinder is shown by a comparison of the buckling loads due to bending between a waffle cylinder and a monocoque cylinder, both of the same weight.

Fundamental Assumptions and Equations

Basic Assumptions

a) The thin shell is integrally stiffened by longitudinal stringers and circumferential rings.

b) The stiffeners are closely and equally spaced such that they are considered continuously distributed over the shell surface.

c) The shell buckles in a mode such that the longitudinal and circumferential half-wave lengths are much greater than the spacing between two adjacent rings and stringers, respectively (i.e., $L/m \gg a$ and $\pi r/n \gg a$).

d) Stiffeners are thin and deep such that uniaxial stress condition prevails. Shear stresses in the directions of the widths of the stiffeners are negligible.

e) Material is linearly elastic and plane cross sections remain plane after bending.

Strain-Displacement Relations

Based on the Love-Kirchhoff assumption that plane cross section remains plane after bending, the strain-displacement relations for the composite stiffened cylindrical shell may be written as

$$\begin{aligned} \varepsilon_1 &= u_x - z w_{xx} \\ \varepsilon_2 &= v_\theta/r - w/r - z(v_\theta + w_{\theta\theta})/r^2 \\ \varepsilon_{12} &= v_x + u_\theta/r - 2z(w_{x\theta} + \tfrac{1}{2}v_x)/r \end{aligned} \tag{1}$$

where the subscripts x and θ refer to the differentiation with respect to x and θ coordinates, respectively.

Stress vs Strain Relations

The stress-strain relations for a homogeneous continuum in a state of plane stress

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} \tag{2}$$

For the shell skin of uniform thickness, the coefficients in Eq. (2) become

$$k_{11} = k_{22} = E/(1-\nu^2) \tag{3a}$$

$$k_{12} = k_{21} = E\nu/(1-\nu^2) = \nu k_{11} \tag{3b}$$

$$k_{33} = E/2(1+\nu) \tag{3c}$$

For the stiffeners or stringers and rings, it is assumed that the closely spaced stiffeners are continuously distributed over the shell surface. The equivalent stress-strain relation for the stiffeners can then be expressed by defining

$$k_{11} = Et_1/a \tag{4a}$$

$$k_{22} = Et_2/a \tag{4b}$$

$$k_{12} = k_{33} = 0 \tag{4c}$$

where t_1/a and t_2/a are the ratios of the stiffener width to the spacing a in the longitudinal and circumferential directions, respectively.

Total Potential Energy

The strain energy expression generally can be described as

$$U = \frac{1}{2}\int_V (\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_{12}\varepsilon_{12})\, dV \tag{5}$$

The strain energy for the shell skin is obtained by substituting Eqs. (1–3) into Eq. (5),

$$\begin{aligned} U_1 = \frac{E}{1-\nu^2}\Bigg[& \frac{1}{2}A\int_0^L\int_0^{2\pi}\Bigg\{u_x{}^2 + 2\nu\left(\frac{u_x v_\theta}{r} - \frac{u_x w}{r}\right) + \\ & \left(\frac{v_\theta{}^2}{r^2} + \frac{w^2}{r^2} - \frac{2v_\theta w}{r^2}\right) + \frac{1-\nu}{2}\left(v_x{}^2 + \frac{u_\theta{}^2}{r^2} + \frac{2u_\theta v_x}{r}\right)\Bigg\} r\, dx\, d\theta + \\ & \frac{1}{2}AZ\int_0^L\int_0^{2\pi}\Bigg\{-2u_x w_{xx} + 2\nu\left(-\frac{v_\theta w_{xx}}{r} + \frac{w w_{xx}}{r} - \frac{u_x v_\theta}{r^2} - \right. \\ & \left.\frac{u_x w_{\theta\theta}}{r^2}\right) - \frac{2v_\theta{}^2}{r^3} + \frac{2v_\theta w}{r^3} - \frac{2v_\theta w_{\theta\theta}}{r^3} + \frac{2w w_{\theta\theta}}{r^3} + \\ & \left(\frac{1-\nu}{2}\right)\left(-\frac{2v_x{}^2}{r} - \frac{2u_\theta v_x}{r^2} - \frac{4v_x w_{x\theta}}{r} - \frac{4u_\theta w_{x\theta}}{r}\right)\Bigg\} r\, dx\, d\theta + \\ & \frac{1}{2}AI\int_0^L\int_0^{2\pi}\Bigg\{w_{xx}{}^2 + \frac{v_\theta{}^2}{r^4} + \frac{w_{\theta\theta}{}^2}{r^4} + \frac{2v_\theta w_{\theta\theta}}{r^4} + \\ & 2\nu\left(\frac{v_\theta w_{xx}}{r^2} + \frac{w_{xx} w_{\theta\theta}}{r^2}\right) + \left(\frac{1-\nu}{2}\right) \times \\ & \left(\frac{v_x{}^2}{r^2} + \frac{4w_{x\theta}}{r^2} + \frac{4v_x w_{x\theta}}{r^2}\right)\Bigg\} r\, dx\, d\theta\Bigg] \end{aligned} \tag{6}$$

where the terms A, I, and Z are integration constants defined as

$$A = \int_{\text{area}} dz, \quad AZ = \int_{\text{area}} z\, dz \quad \text{and} \quad AI = \int_{\text{area}} z^2\, dz \tag{7}$$

The strain energy for the stiffeners is obtained by substituting Eqs. (1), (2), and (4) into Eq. (5),

$$\begin{aligned} U_2 = \frac{E}{a}\Bigg[& \frac{A}{2}\int_0^L\int_0^{2\pi}\left\{t_1 u_x{}^2 + t_2\left(\frac{v_\theta{}^2}{r^2} + \frac{w^2}{r^2} - \frac{2v_\theta w}{r^2}\right)\right\} r\, dx\, d\theta + \\ & \frac{AZ}{2}\int_0^L\int_0^{2\pi}\Bigg\{-2t_1 u_x w_{xx} + t_2\left(-\frac{2v_\theta{}^2}{r^3} + \frac{2v_\theta w}{r^3} - \right. \\ & \left.\frac{2v_\theta w_{\theta\theta}}{r^3} + \frac{2w w_{\theta\theta}}{r^3}\right)\Bigg\} r\, dx\, d\theta + \\ & \frac{AI}{2}\int_0^L\int_0^{2\pi}\left\{t_1 w_{xx}{}^2 + t_2\left(\frac{v_\theta{}^2}{r^4} + \frac{w_{\theta\theta}{}^2}{r^4} + \frac{2v_\theta w_{\theta\theta}}{r^4}\right)\right\} r\, dx\, d\theta\Bigg] \end{aligned} \tag{8}$$

The potential energy of the applied middle-surface loads is defined as the work done by such loads during the deformation of the waffle shell

$$W = \int_0^L\int_0^{2\pi} N_x(u_x + \tfrac{1}{2}w_x{}^2 + \tfrac{1}{2}v_x{}^2 + e w_{xx})\, r\, dx\, d\theta \tag{9}$$

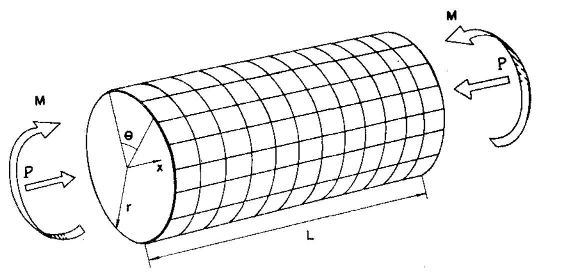

Fig. 2 An orthogonally stiffened waffle cylindrical shell under pure bending ($N_b = M/\pi R^2$; $N_c = P/2\pi R$).

where N_x is the force per unit length in the x direction; e is the eccentricity of the load N_x from the reference (middle) surface of the skin. The total potential energy of the waffle shell system is thus

$$\Pi = U_1 + U_2 - W \tag{10}$$

The principle of minimum potential energy states that, of all displacements satisfying given boundary conditions for a system, those which satisfy the equilibrium conditions make the potential energy Π assume a stationary value. For stable equilibrium, Π is a minimum and the first variation of Π vanishes, i.e., $\delta\Pi = 0$. When the system is perturbed from a stable equilibrium to an unstable state, it passes through a neutral stability state for which the second variation of the potential energy vanishes, i.e., $\delta^2\Pi = 0$. The buckling criterion is deduced from such neutral equilibrium state.

Formulation for the Buckling Analysis of Cylindrical Shell with Simply Supported Edges Under Uniform Compression

For a cylindrical waffle shell with simply-supported boundary conditions, the displacements may be represented as

$$\begin{aligned} u &= \sum_{m=1}^{k}\sum_{n=0}^{l-1} U_{mn}\cos\frac{m\pi x}{L}\cos n\theta \\ v &= \sum_{m=1}^{k}\sum_{n=0}^{l-1} V_{mn}\sin\frac{m\pi x}{L}\sin n\theta \\ w &= \sum_{m=1}^{k}\sum_{n=0}^{l-1} W_{mn}\sin\frac{m\pi x}{L}\cos n\theta \end{aligned} \tag{11}$$

where m and n are integers; U_{mn}, V_{mn}, and W_{mn} are unknown displacement amplitudes; and integers k and l are the numbers of functions required to represent accurately the u, v, and w displacements.

After substituting the foregoing displacement functions into the expression for total potential energy Π [Eqs. (6), (8), and (9)] and then executing the first variation of Π, Eq. (10), with respect to each of the displacement amplitudes, U_{mn}, V_{mn}, W_{mn}, the equilibrium equations are obtained. Such procedure was given in detail in Ref. 2 in the stability analysis of a waffle plate. One thus has

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & (C_{22}-\lambda) & C_{23} \\ C_{31} & C_{22} & (C_{33}-\lambda) \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} F_{1_{mn}} \\ F_{2_{mn}} \\ F_{3_{mn}} \end{Bmatrix} \tag{12}$$

with C_{ij}'s for internally stiffened shell are as follows:

$$C_{11} = \frac{Eh}{1-\nu^2}\left\{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{1-\nu}{2}\right)\left(\frac{n}{r}\right)^2\right\} + \frac{Et_1 d}{a}\left(\frac{m\pi}{L}\right)^2$$

$$C_{12} = C_{21} = -\frac{Eh}{1-\nu^2}\left\{\nu\left(\frac{m\pi}{L}\right)\left(\frac{n}{r}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{m\pi}{L}\right)\left(\frac{n}{r}\right)\right\}$$

$$C_{13} = C_{31} = \frac{\nu Eh}{1-\nu^2}\cdot\frac{1}{r}\left(\frac{m\pi}{L}\right) - \frac{Et_1 d(d+h)}{2a}\left(\frac{m\pi}{L}\right)^3$$

$$C_{22} = \frac{Eh}{1-\nu^2}\left\{\left(\frac{n}{r}\right)^2 + \left(\frac{1-\nu}{2}\right)\left(\frac{m\pi}{L}\right)^2\right\} + \frac{Eh^3}{12(1-\nu^2)}\frac{1}{r^2} \times \left\{\left(\frac{n}{r}\right)^2 + \left(\frac{1-\nu}{2}\right)\left(\frac{m\pi}{L}\right)^2\right\} + \frac{Et_2 d}{a}\left(\frac{n}{r}\right)^2 - \frac{Et_2 d(d+h)}{a} \times \frac{1}{r}\left(\frac{n}{r}\right)^2 + \frac{Et_2 d}{a}\left(\frac{d^2}{3} + \frac{hd}{2} + \frac{h^2}{4}\right)\cdot\frac{1}{r^2}\left(\frac{n}{r}\right)^2$$

$$C_{23} = C_{32} = -\frac{Eh}{1-\nu^2}\cdot\frac{1}{r}\left(\frac{n}{r}\right)^2 - \frac{Eh^3}{12(1-\nu^2)}\cdot\frac{1}{r} \times \left\{\nu\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{r}\right) + \left(\frac{n}{r}\right)^3 + (1-\nu)\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{r}\right)\right\} - \frac{Et_2 d}{a}\cdot\frac{1}{r}\left(\frac{n}{r}\right) + \frac{Et_2 d(d+h)}{2a}\cdot\frac{1}{r^2}\left\{\left(\frac{n}{r}\right) + r^2\left(\frac{n}{r}\right)^3\right\} - \frac{Et_2 d}{a}\left(\frac{d^2}{3} + \frac{hd}{2} + \frac{h^2}{4}\right)\cdot\frac{1}{r}\left(\frac{n}{r}\right)^3$$

$$C_{33} = \frac{Eh}{1-\nu^2}\cdot\frac{1}{r^2} + \frac{Eh^3}{12(1-\nu^2)}\left\{\left(\frac{m\pi}{L}\right)^4 + 2\nu\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{r}\right)^2 + \left(\frac{n}{r}\right)^4 + 2(1-\nu)\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{r}\right)^2\right\} + \frac{Et_2 d}{a}\cdot\frac{1}{r^2} - \frac{Et_2 d(d+h)}{a} \times \frac{1}{r}\left(\frac{n}{r}\right)^2 + \frac{Ed}{a}\left(\frac{d^2}{3} + \frac{hd}{2} + \frac{h^2}{4}\right)\left\{t_1\left(\frac{m\pi}{L}\right)^4 + t_2\left(\frac{n}{r}\right)^4\right\} \tag{12a}$$

and c_{ij}'s for externally stiffened shell are as follows:

$$C_{11} = \frac{Eh}{1-\nu^2}\left\{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{1-\nu}{2}\right)\left(\frac{n}{r}\right)^2\right\} + \frac{Et_1 d}{a}\left(\frac{m\pi}{L}\right)^2$$

$$C_{12} = C_{21} = -\frac{Eh}{1-\nu^2}\left\{\nu\left(\frac{m\pi}{L}\right)\left(\frac{n}{r}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{m\pi}{L}\right)\left(\frac{n}{r}\right)\right\}$$

$$C_{13} = C_{31} = \frac{\nu Eh}{1-\nu^2}\cdot\frac{1}{r}\left(\frac{m\pi}{L}\right) + \frac{Et_1 d(d+h)}{2a}\left(\frac{m\pi}{L}\right)^3$$

$$C_{22} = \frac{Eh}{1-\nu^2}\left\{\left(\frac{n}{r}\right)^2 + \left(\frac{1-\nu}{2}\right)\left(\frac{m\pi}{L}\right)^2\right\} + \frac{Eh^3}{12(1-\nu^2)}\frac{1}{r^2} \times \left\{\left(\frac{n}{r}\right)^2 + \left(\frac{1-\nu}{2}\right)\left(\frac{m\pi}{L}\right)^2\right\} + \frac{Et_2 d}{a}\left(\frac{n}{r}\right)^2 + \frac{Et_2 d(d+h)}{a} \times \frac{1}{r}\left(\frac{n}{r}\right)^2 + \frac{Et_2 d}{a}\left(\frac{d^2}{3} + \frac{hd}{2} + \frac{h^2}{4}\right)\cdot\frac{1}{r^2}\left(\frac{n}{r}\right)^2$$

$$C_{23} = C_{32} = -\frac{Eh}{1-\nu^2}\cdot\frac{1}{r}\left(\frac{n}{r}\right)^2 - \frac{Eh^3}{12(1-\nu^2)}\cdot\frac{1}{r}\left\{\nu\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{r}\right) + \left(\frac{n}{r}\right)^3 + (1-\nu)\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{r}\right)\right\} - \frac{Et_2 d}{a}\cdot\frac{1}{r}\left(\frac{n}{r}\right) - \frac{Et_2 d(d+h)}{2a}\cdot\frac{1}{r^2}\left\{\left(\frac{n}{r}\right) + r^2\left(\frac{n}{r}\right)^3\right\} - \frac{Et_2 d}{a}\left(\frac{d^2}{3} + \frac{hd}{2} + \frac{h^2}{4}\right)\cdot\frac{1}{r}\left(\frac{n}{r}\right)^3$$

$$C_{33} = \frac{Eh}{1-\nu^2}\cdot\frac{1}{r^2} + \frac{Eh^3}{12(1-\nu^2)}\left\{\left(\frac{m\pi}{L}\right)^4 + 2\nu\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{r}\right)^2 + \left(\frac{n}{r}\right)^4 + 2(1-\nu)\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{r}\right)^2\right\} + \frac{Et_2 d}{a}\cdot\frac{1}{r^2} + \frac{Et_2 d(d+h)}{a}\cdot\frac{1}{r}\left(\frac{n}{r}\right)^2 + \frac{Ed}{a}\left(\frac{d^2}{3} + \frac{hd}{2} + \frac{h^2}{4}\right)\left\{t_1\left(\frac{m\pi}{L}\right)^4 + t_2\left(\frac{n}{r}\right)^4\right\} \tag{12b}$$

and $\lambda = N_c(m\pi/L)^2$, where $\{U_{mn}\}$, $\{V_{mn}\}$, $\{W_{mn}\}$ and $\{F_{1_{mn}}\}$, $\{F_{2_{mn}}\}$, $\{F_{3_{mn}}\}$ are the displacement amplitudes and the corresponding generalized forces. Since the values of m and n vary from 1 to k and from 0 to $(l-1)$ respectively, the size of the matrix in Eq. (11) is $3kl$ by $3kl$.

By vanishing the second variation of the total potential energy, the state of neutral stability is obtained. Such procedure results in vanishing the determinant of the stiffness matrix,

$$\det \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & (C_{22}-\lambda) & C_{23} \\ C_{31} & C_{32} & (C_{33}-\lambda) \end{vmatrix} = 0 \quad (13)$$

It is interesting to note that, for the particular case in which the axial compression is uniformly distributed, Eq. (13) becomes kl sets of uncoupled submatrices with the same order 3×3. All these 3×3 submatrices are also in the same form as described by Eq. (12a). The solution for the critical buckling load is therefore reduced to the search for the smallest eigenvalue λ in the 3×3 matrix equation for various values of m and n.

This uncoupling effect can also be interpreted physically. Since in the present case the effect of the stiffeners are averaged out through the shell surface, the buckling mode of the shell when subjected to uniform axial compression becomes the same as that for the monocoque shell. It is well known in that case[1] that the buckling mode is represented by only one mode with a pair of particular values of m and n.

In some application, the waffle shells are further strengthened by adding some discrete type primary stiffeners (see, e.g., Ref. 2). With the effect of such discrete stiffeners, the buckling mode shape can only be represented accurately by the summations of trigonometric functions as defined in Eq. (11) even for the case of uniform compression. The stiffness coefficient due to the effect of such discrete stiffeners can be formulated quite straightforwardly[2] and added to the matrix equation (12). In that case, the coefficients in Eq. (12) become coupled with one another and a large matrix of order $3kl$ by $3kl$ must be solved.

Formulation for Buckling Analysis of Cylindrical Shell with Simply Supported Edges under Combined Bending and Axial Loads

When a cylindrical waffle shell is subjected to a pair of pure bending moments and axial loads applied at both ends, as shown in Fig. 2, the distribution of axial stress along the reference surface in the skin may be described as

$$N_x = N_b \cos\theta + N_c \quad (14)$$

where N_b is the maximum bending stress and N_c is the uniform axial stress.

For the case of simply supported edges, the displacement functions may be written as

$$u = \cos\frac{m\pi x}{L}\sum_{n=0}^{l-1} U_{mn}\cos n\theta \quad (15a)$$

$$v = \sin\frac{m\pi x}{L}\sum_{n=0}^{l-1} V_{mn}\sin n\theta \quad (15b)$$

$$w = \sin\frac{m\pi x}{L}\sum_{n=0}^{l-1} W_{mn}\cos n\theta \quad (15c)$$

Such displacement functions have been used previously for the case of monocoque shells.[5]

By the use of the displacement functions given in Eq. (15) and the stress distribution given by Eq. (14), an equation of equilibrium can be derived by performing the similar procedure as described in the previous section. One thus has

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33}-\lambda \end{bmatrix}\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ \psi \end{Bmatrix} = \begin{Bmatrix} F_{1_{mn}} \\ F_{2_{mn}} \\ F_{3_{mn}} \end{Bmatrix} \quad (16)$$

with

$$\psi = \frac{N_o}{2}\left(\frac{m\pi}{L}\right)^2\left\{\frac{(1+\delta_{1n}-\delta_{0n})}{(1+\delta_{0n})}W_{m(n-1)} + W_{m(n+1)}\right\} \quad (17)$$

where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, and $\lambda = N_c(m\pi/L)^2$, in which all the parameters were in the previous section. It is noted that, as a common practice during the derivation of Eq. (17), the term $v_x^2/2$ has been neglected from Eq. (9) for its smallness. The buckling loads N_b or eigenvalues are obtained by the neutral stability condition that the determinant of the $3l$ by $3l$ stiffness coefficient matrix in Eq. (16) be equal to zero.

Application

The application of the foregoing derivation was demonstrated through a variety of examples.

Buckling of Waffle Cylinder Under Uniform Compression

The first example is the analysis of buckling load of an internally stiffened cylindrical shell with simply supported edges and subjected to uniform compression N_x. The dimensions and the Young's modulus are defined as: $L = 50$ in.; $r = 48$ in.; $h = 0.05$ in.; $t_1 = t_2 = 0.125$ in.; $d_1 = d_2 = 0.2$ in.; $a = 3$ in.; and $E = 10.6 \times 10^6$ psi. The definitions of these dimensional parameters are defined in Fig. 1.

During the formulation, the effect of the stiffeners is averaged out over the shell surface, and the buckling mode of the shell becomes the same as that for the case of monocoque shell. In each of the displacement functions defined in Eq. (11), only one term instead of a summation of terms is required. The stiffness matrix equation (12) reduces to an order of 3 by 3 with m, n, and λ as unknown parameters. The solution of buckling load for such eigenvalue equations involves a search of the wave numbers m and n which result in a minimum eigenvalue.

In this study, a search of minimum eigenvalue was made with m varied from 1 to 20 and n from 0 to 20. The results can be plotted in the form of festoon curves. For clarity of presentation, only a few selected curves are presented. The lowest buckling load occurs at $m = 6$ and $n = 17$ with the value of 789.82 lb/in.

The present example was selected because Meyer[1] has analyzed the same example previously with the differences that his stiffeners were assumed to be oriented at 45° with the generator and that the cylinder was subjected to a slight internal pressure of 2.1 psi. It is interesting to note that Meyer has obtained a buckling load of 792 lb/in., which is almost identical to the present value of 789.82 lb/in. This seems to imply that both ways of stiffening (90° and 45° orientations) may result in the same buckling strength. A monocoque cylinder with the same weight and the same values for length and radius also was analyzed. The buckling load was found to be 594.86 lb/in. It is thus seen that an increase of buckling strength by 32.8% is achieved by wall stiffening yet without increasing the total weight.

Optimization Analysis of Buckling of a Constant Weight Waffle Cylinder

Since the minimum weight analysis of waffled cylindrical shell under compressive loading is of fundamental interest in the structural design of launch and space vehicles, the present study has included examples of optimization analysis of the buckling of a simply supported constant weight cylinder subjected to uniform axial compression.

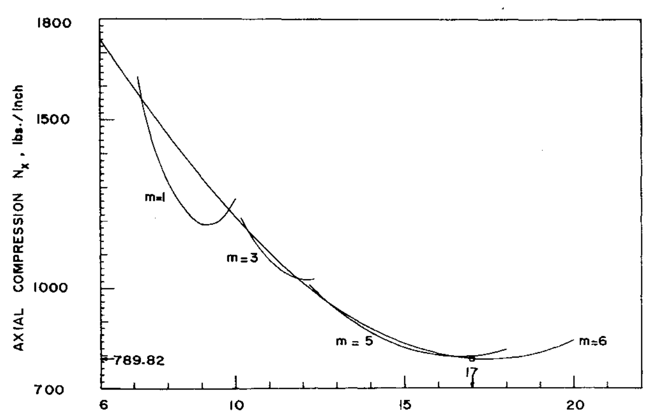

Fig. 3 Festoon curves for the search of lowest eigenvalue (buckling load).

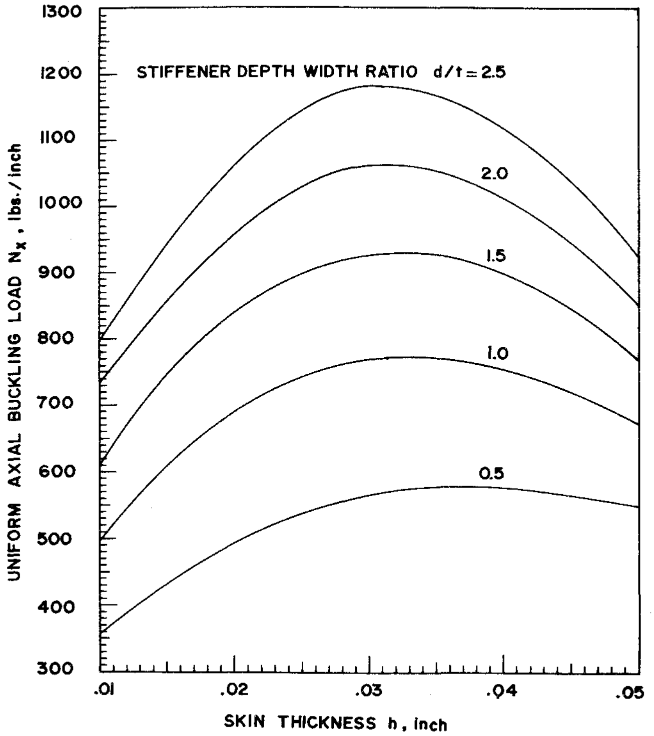

Fig. 4 Optimization analysis of the buckling of a constant weight waffle cylinder (101.54 lb) with skin thickness and stiffener depth-width ratio as variables ($a = 3$ in.; $t_1 = t_2 = t$; $r = 48$ in.; $L = 50$ in.).

In the first example, the skin thickness h and the stiffener depth-width ratio d/t were varied while the other parameters were held constant: $a = 3$ in.; $t_1 = t_2 = t$; $r = 48$ in.; $L = 50$ in.; and total weight = 101.54 lb (for aluminium mass density = 0.101 lb/in.^{3}). The results for the buckling load were plotted in Fig. 4. Every point in Fig. 4 involved a search for a set of numbers for m and n that yielded the lowest eigenvalue. It is seen in Fig. 4 that by increasing the depth-width ratio of the stiffener the buckling strength of the cylinder may be increased considerably. Also, the buckling load curve of every depth-width ratio yields a maximum which occurs for the skin thickness in the region of 0.03 to 0.035 in. Because of manufacturing difficulties and aerodynamic considerations, the depth-width ratio of the stiffeners cannot be increased indefinitely. For a feasible value of depth-width ratio, a value of skin thickness that gives the optimum strength may exist. In this particular example, it is around 0.03 in.

In the second example, the skin thickness h and the width ratio between stringers and rings t_1/t_2 were varied while the other parameters were held constant: $a = 3$ in.; $d_1 = d_2 = 0.20$ in.; $r = 48$ in.; $L = 50$ in.; and total weight = 101.54 lb. The results for the buckling load were plotted in Fig. 5. It is seen that by increasing the skin thickness the buckling strength of the cylinder may be increased considerably. In this case, the width-ratio t_1/t_2 should not be chosen smaller than 0.6. When t_1/t_2 is greater than 0.6, the variation of width ratio seems to affect the buckling load slightly, and a maximum strength occurs when the width ratio t_1/t_2 is equal to 1. It is noted that the skin cannot be made too thin due to manufacturing and fabricational difficulties. The initial imperfection could affect the buckling strength considerably.

To conclude the results obtained in Figs. 3–5, it is seen that a highest buckling load of 1179.51 lb/in. could be obtained with $L = 50$ in.; $r = 48$ in.; $h = 0.03$ in.; $t_1 = t_2 = 0.148$ in.; $d_1 = d_2 = 0.370$ in. When compared to the 789.82 lb/in. obtained for the waffle cylinder in the first example with the same total weight, an increase of buckling strength by 49.4% is indicated. When compared to the 594.86 lb/in. obtained for the monocoque cylinder with the same weight, an increase of buckling strength by 98.2% is noted.

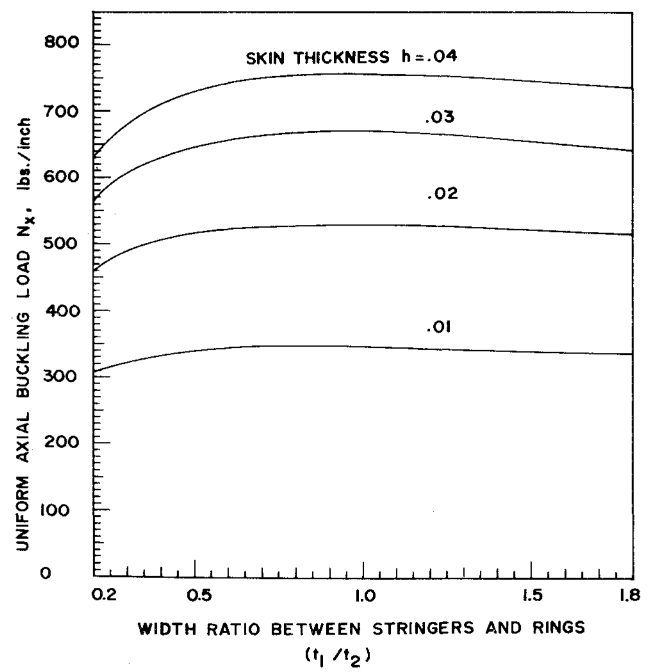

Fig. 5 Optimization analysis of the buckling of a constant weight cylinder (101.54 lb) with skin thickness and width ratio between stringers and rings as variables ($a = 3$ in.; $d_1 = d_2 = 0.2$ in.; $r = 48$ in.; $L = 50$ in.).

Buckling of a Waffle Cylinder Under Pure Bending

For the case of simply supported waffle cylinder under combined bending and axial loads, the formulation for buckling analysis is presented in Eq. (16). By setting the width and the depth of the stiffeners equal to zero, the formulation can be used to analyze the buckling of a simply supported monocoque cylinder under pure bending ($N_c = 0$). The latter case has been investigated previously by Seide and Weingarten[5] by the use of Galerkin's method combined with Batdorf's modified version of Donnell's equation.

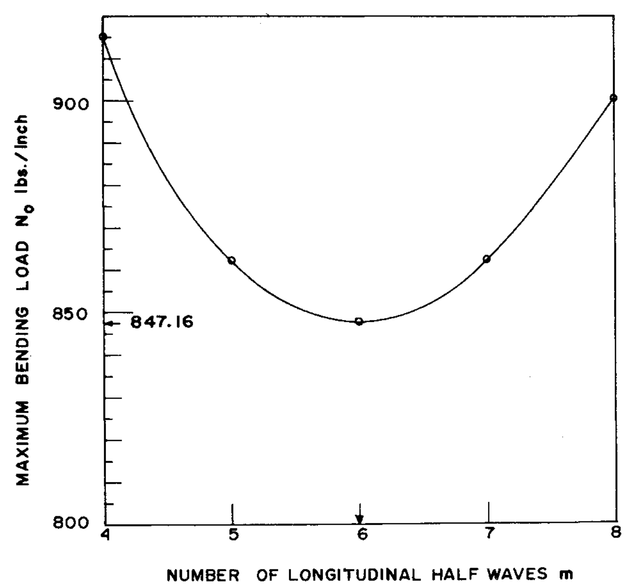

Fig. 6 Variation of maximum buckling bending stress N_b with longitudinal half wave number m for a simply supported waffle cylinder under pure bending.

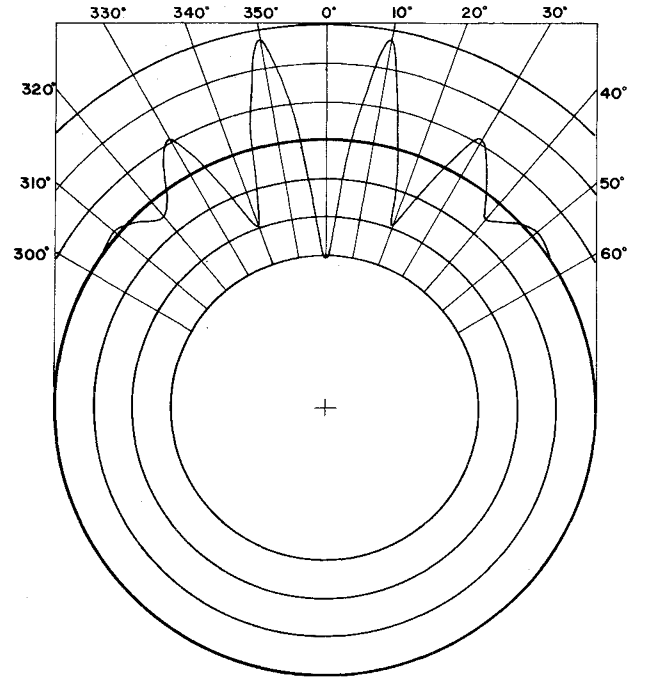

Fig. 7 Circumferential buckling mode of the simply supported waffle cylinder under pure bending (the displacement is magnified 6.857 times).

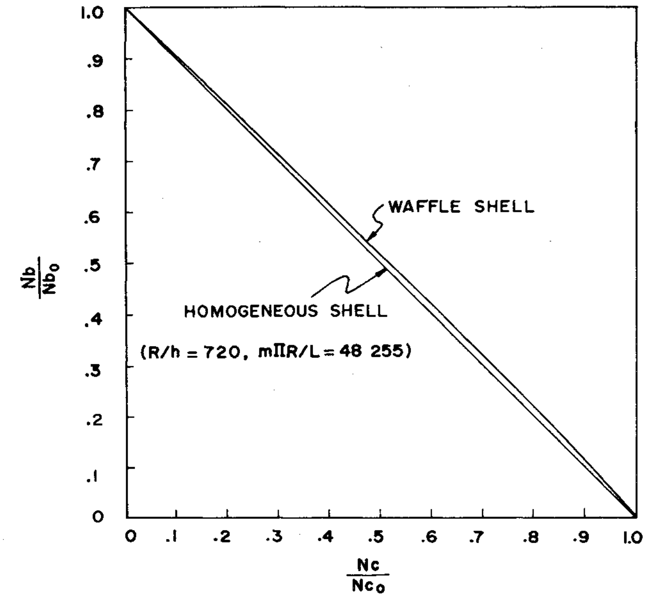

Fig. 8 Interaction of critical bending stress N_b/N_{bo} and critical uniform axial stress N_c/N_{co}.

The first example chosen in this study is a monocoque simply-supported cylindrical shell subjected to pure bending with $L = 50$ in.; $r = 48$ in.; $h = 0.0667$ in., and weight = 101.54 lbs. A total of 28 terms were used in the displacement series as assumed in Eq. (15). The maximum bending buckling stress N_b was found to be 603.58 lb/in. with longitudinal half-wave number m to be 16. The Seide and Weingarten[5] procedure also was used to analyze this problem. Identical results were obtained.

A simply supported waffle cylinder under pure bending was next analyzed. The length L, radius r, and the weight are the same as that of the monocoque cylinder. The other dimensions were assumed to be $t_1 = t_2 = 0.125$ in.; $d_1 = d_2 = 0.2$ in. This is in fact the same cylinder as that in the first example. A total of 28 terms were employed in the displacement series, as assumed in Eq. (15). The results for the buckling load for various values of m were found and plotted in Fig. 6. The lowest eigenvalue or the critical buckling load occurs at $m = 6$ with a value of maximum bending stress equal to 847.16 lb/in. The circumferential buckling mode shape was plotted in Fig. 7. The amplitudes of the buckling waves in the compressive zone varies with the magnitude of the compressive bending stress. The amplitudes of the buckling waves in the tensile zone are too small to be shown.

When comparing the maximum bending stress for the monocoque cylinder to that for the waffle cylinder, both weighing 101.54 lb, it is seen that the latter is greater than the former by 40.36%. It is obvious that the bending buckling strength of the cylinder can be increased considerably by wall stiffening yet without increasing the total weight. It is of interest to note that the maximum buckling bending stress for the simply supported waffle cylinder is 1.075 times that of the uniform compressive buckling stress. This ratio is quite close to the value of unity found for the case of homogeneous cylinder.

An interaction relationship between the uniform axial compressive stress and bending stress also was studied. A plot of the ratio between the critical bending stress to the maximum critical bending stress N_b/N_{bo} vs the ratio between the critical compressive stress to the maximum critical compressive stress N_c/N_{co} was shown in Fig. 8 for both homogeneous and waffle cylindrical shells. Both cylinders exhibit the similar behavior. It is noted that the relation for the case of homogeneous cylinder is identical to that obtained by Seide and Weingarten.[5]

Conclusions

The energy expressions for a waffle cylindrical shell with integrally milled orthogonal stiffeners have been derived. The principle of minimum potential energy has been employed to derive the stiffness matrix equations for buckling analysis for a simply supported waffle cylinder under both uniform and linearly distributed compressive stresses.

The formulations and solution procedures have been demonstrated through application to a variety of examples. For the case of uniform compressive stress, the total weight of the cylinder was held constant but the skin thickness and the stiffener sizes were varied. Some conclusions have been drawn with regard to the optimization of these parameters so that the maximum buckling strength can be obtained. The fact that considerable buckling strength can be obtained by wall stiffening yet without increasing weight has been demonstrated with numerical evidence.

For a particular example of simply supported cylindrical shell under pure bending, it was found that the buckling strength of the monocoque cylinder can be increased by 40.36% by certain types of wall stiffening. It was also found that the maximum buckling bending stress for the simply-supported waffle cylinder considered is 1.075 times of the uniform compressive buckling stress.

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